ON THE ERROR TERM OF A LATTICE COUNTING PROBLEM

OLIVIER BORDELLÈS, FLORIAN LUCA, AND IGOR E. SHPARLINSKI

Abstract. We improve the error terms of some estimates related to counting lattices from recent work of L. Fukshansky, P. Guerzhoy and F. Luca (2017). This improvement is based on some analytic techniques, in particular on bounds of exponential sums coupled with the use of Vaaler polynomials.

1. Introduction

1.1. **Background.** For integer $T \geq 1$, we let

$$\mathcal{F}(T) = \{a/b \ : \ (a,b) \in \mathbb{Z}^2, \ 0 \leq a < b \leq T, \ \gcd(a,b) = 1\}$$

be the set of Farey fractions. We also define

$$\mathcal{I}(T) = \mathcal{F}(T) \cap [0, 1/2] \, .$$

Now, following [6], we consider the quantity

$$C(T) = \sum_{a/b \in \mathcal{I}(T)} \# \mathcal{C}_{a,b}(T),$$

where

$$\mathcal{C}_{a,b}(T) = \mathcal{F}(T) \cap [1 - a^2/b^2, 1].$$

The quantity $C(T)$ appears naturally in some counting problems for two-dimensional lattices. More precisely, every similarity class of *planar lattices* can be parametrised by a point $\tau = x_0 + iy_0$ in

$$\mathcal{R} = \{\tau = x_0 + iy_0 \ : \quad 0 \leq x_0 \leq 1/2, \quad y_0 \geq 0, \quad |\tau| \geq 1\} \subseteq \mathbb{C},$$

where one identifies $\tau \in \mathcal{R}$ with the lattice

$$\Lambda_\tau = \begin{pmatrix} 1 & x_0 \\ 0 & y_0 \end{pmatrix} \mathbb{Z}^2.$$

Further, similarity classes of *arithmetic* planar lattices correspond to Λ_τ, where

$$\tau = a/b + i\sqrt{c/d}$$

2010 *Mathematics Subject Classification.* Primary 11H06, 11K38, 11P21.
Key words and phrases. Lattices, Farey fractions, Vaaler polynomials.

for integers a, b, c, d such that

$$\gcd(a,b) = \gcd(c,d) = 1, \quad 0 \le a \le b/2, \quad d > 0, \quad c/d \ge 1 - a^2/b^2.$$

The class is *semistable* if furthermore $c \le d$. With these conventions, the quantity $C(T)$ counts the number of similarity classes of semistable arithmetic planar lattices of height at most T, that if for which $\max\{a, b, c, d\} \le T$.

1.2. **Previous results and motivation.** The following result appears as [6, Lemma 3.2]:

$$C(T) = \frac{3}{8\pi^4} T^4 + O(T^3 \log T). \tag{1.1}$$

Our goal here is to sharpen the error term in the asymptotic formula (1.1). Furthermore, some of motivation comes from the intention to introduce into the investigation of this question several results and techniques of analytic number theory, which have not been employed in the derivation of (1.1).

Besides, our approach leads us to some interesting and apparently new questions about Farey fractions, which we address here as well.

1.3. **Our results.** Here we give a direct improvements of (1.1) and show that that error term can be reduced by about $(\log T)^{1/3+o(1)}$ (see Corollary 1.3 below for a precise result). However, it seems to be more natural to express the main term via some general quantities related to Farey fractions and then try to minimize the error term. In particular, we outline some results on counting Farey fractions in Section 2.2.

Here, we accept this point of view and thus express the main term of the asymptotic formula for $\#C(T)$ via the cardinality

$$F(T) = \#\mathcal{F}(T)$$

of the set of of Farey fractions and also second moment of the Farey fractions in $[0, 1/2]$:

$$G(T) = \sum_{\substack{\xi \in \mathcal{F}(T) \\ \xi \le 1/2}} \xi^2, \qquad \nu = 0, 1, \ldots.$$

It is also convenient to define

$$M(t) = \sum_{1 \le k \le t} \mu(k). \tag{1.2}$$

As usual $A = O(B)$, $A \ll B$, $B \gg A$ are equivalent to $|A| \le c|B|$ for some *absolute* constant $c > 0$, whereas $A = o(B)$ means that $A/B \to 0$.

Theorem 1.1. *We have*

$$C(T) = F(T)G(T) + O\left(T^{11/4+o(1)} + T^3\delta(T^{1/2})\log T\right),$$

where $\delta(t)$ is any decreasing function such that

$$|M(t)| \le t\delta(t)$$

holds.

By the classical bound of Walfisz [23, Chapter V, Section 5, Equation (12)] one can take

$$\delta(t) = \exp(-c(\log t)^{3/5}(\log\log t)^{-1/5}) \tag{1.3}$$

for absolute constant $c > 0$, hence immediately producing the bound $O(T^3\exp(-c_0(\log T)^{3/5}(\log\log T)^{-1/5}))$ for some constant $c_0 > 0$ on the error term in Theorem 1.1.

In (2.8) below we obtain an approximation to $G(T)$ via $F(T)$ which implies the following result.

Corollary 1.2. *We have*

$$C(T) = \frac{1}{24}F(T)^2 + O(T^3).$$

Finally, using the asymptotic formula for $F(T)$ with the error term given by (2.4), we obtain the following direct improvement of (1.1):

Corollary 1.3. *We have*

$$C(T) = \frac{3}{8\pi^4}T^4 + O(T^3(\log T)^{2/3}(\log\log T)^{4/3})$$

as $T \to \infty$.

Under the Riemann Hypothesis, we can take [1, 20]

$$\delta(t) = t^{-1/2}\rho(t), \tag{1.4}$$

where

$$\rho(t) = \exp\left((\log t)^{1/2}(\log\log t)^{5/2+o(1)}\right). \tag{1.5}$$

Without $\rho(t)$, the inequality (1.4) is known as a conjecture of Mertens which has been refuted by Odlyzko and te Riele [17].

In particular, under the Riemann Hypothesis, the error term of Theorems 1.1 becomes $T^{11/4+o(1)}$. However, in this case a different approach leads to a better result.

Theorem 1.4. *Assume the Riemann Hypothesis. Then*

$$C(T) = F(T)G(T) + O\left(T^{752/283}\rho(T)\log T\right),$$

where ρ is defined in (1.5) above.

Note that

$$\frac{11}{4} = 2.75, \qquad \text{and} \qquad \frac{752}{283} = 2.65724\ldots,$$

and the proofs of both Theorems 1.1 and 1.4 are based on bounds of exponential sums, however of quite different shape in each of proofs. In particular, Theorem 1.4 is a based on an application of a new *exponent pair*

$$(k, \ell) = \left(\frac{13}{84}, \frac{55}{84}\right) \tag{1.6}$$

due to Bourgain [2]. Although we have taken care of minimising the power of T in Theorem 1.4 it is quite possible that with more thorough optimisation one can get a slight improvement of this power. It is also interesting to note that the classical exponent pair $(1/2, 1/2)$ of van der Corput combined with the optimisation algorithm of Graham and Kolesnik [7, Chapter 5] lead to a slightly higher value of the power of T, namely to

$$2R + 1 = 2.65804\ldots,$$

where R is the so-called *Ranking constant*, see [7, Section 5.4]. We also note under the *exponent pair conjecture* that for any $\varepsilon > 0$, the pair $(\varepsilon, 1/2 + \varepsilon)$ is admissible, the result of Theorem 4.2 below leads to the error term $O(T^{5/2+o(1)})$ in Theorem 1.4.

We remark that improving the error term in Corollary 1.3 is probably impossible until the bound (2.4) below is improved. However, it is plausible that one can improve (2.8) and thus obtain a stronger version of Corollary 1.2, which we pose as an open question.

2. Main Term

2.1. **Initial transformations.**
Our treatment of the main term is the same for Theorems 1.1 and 1.4.

By a result of Niederreiter [15], for any integers $0 \le a < b$ the following formula holds

$$\#\mathcal{C}_{a,b}(T) - \frac{a^2}{b^2} F(T) = -\sum_{n=1}^{T} \sum_{d \mid n} \mu(n/d) \{da^2/b^2\}, \tag{2.1}$$

where $\mu(k)$ is the Möbius function (see [9, Equation (1.16)]) and $\{\alpha\}$ is the fractional part of a real α.

We rewrite (2.1) as

$$\#\mathcal{C}_{a,b}(T) - \frac{a^2}{b^2} F(T) = -\sum_{d=1}^{T} \{da^2/b^2\} M(T/d),$$

where $M(t)$ is given by (1.2). We now write

$$C(T) - \mathfrak{M}(T) = \mathfrak{E}(T), \tag{2.2}$$

where

$$\begin{aligned} \mathfrak{M}(T) &= F(T) \sum_{a/b \in \mathcal{I}(T)} \frac{a^2}{b^2} = F(T)G(T), \\ \mathfrak{E}(T) &= - \sum_{a/b \in \mathcal{I}(T)} \sum_{d=1}^{T} \{da^2/b^2\} M(T/d). \end{aligned} \tag{2.3}$$

Using either of the bounds (1.3) and (1.4) gives the bound $O(T)$ for each inner sum in the definition of the error term $\mathfrak{E}(T)$ (see, for example, the proof of [15, Lemma 2]), and thus yields the conclusion of Theorem 1.1 with an error term $O(T^3)$. Thus, to do better, we need to investigate the *cancellations* between these sums.

2.2. **Counting Farey fractions.** Here, we collect some known facts about Farey fractions.

The set $\mathcal{F}(T)$ has been the subject of a lot of research. Writing $\varphi(n)$ for the Euler function of the positive integer n, we have

$$F(T) = \sum_{b \le T} \varphi(b) = \frac{3}{\pi^2} T^2 + R(T).$$

The error term $R(T)$ above has also been the subject of a lot of research. For example, by the classical result of Mertens [13] (that dates back to 1874), we have

$$R(T) = O(T \log T).$$

This has been improved by Walfisz [23, Chapter V, Section 5, Equation (35)]

$$R(T) = O\left(T (\log T)^{2/3} (\log\log T)^{4/3}\right) \tag{2.4}$$

as $T \to \infty$. Note that Saltykov [19] has announced a slightly stronger result, however the proof appears to be wrong, see [18].

Erdős and Shapiro [5] have shown that

$$R(T) = \Omega_{\pm}(T \log\log\log\log T),$$

which means that for some positive constant c, each of the inequalities

$$R(T) > cT \log\log\log\log T \qquad \text{and} \qquad R(T) < -cT \log\log\log\log T$$

holds infinitely often, while Montgomery [14] has sharpened this to

$$R(T) = \Omega_{\pm}(T(\log\log T)^{1/2}).$$

Average values and moments of $R(T)$ have also been considered. For example,

$$\sum_{m\le T} R(m) = \frac{3T^2}{2\pi^2} + O(T^2\eta(T)) \tag{2.5}$$

(see [21]), and

$$\sum_{m\le T} R(m)^2 = \left(\frac{1}{6\pi^2} + \frac{2}{\pi^4}\right) T^4 + O(T^3\eta(T)), \tag{2.6}$$

(see [4]), where in both (2.5) and (2.6)

$$\eta(T) = \exp(-A(\log T)^{3/5}(\log\log T)^{-1/5})$$

for some constant $A > 0$ (not necessarily the same one in both (2.5) and (2.6)).

We remark that for the second (and other) moments of Farey fractions one can obtain asymptotic formulas via the general bounds on the difference between sums of continuous functions on Farey fractions and the corresponding integrals (see [3, 24]).

Unfortunately, these results do not seem to apply to the sum $G(T)$. On the other hand, one can, via elementary but rather tedious arguments, relate $G(T)$ to $F(T)$ and then show that

$$G(T) = \frac{1}{8\pi^2}T^2 + O\left(T(\log T)^{2/3}(\log\log T)^{4/3}\right) \tag{2.7}$$

as $T \to \infty$. However, here we use some general results to derive (2.7). We start with recalling the bound

$$\Delta(T) = O(T^{-1})$$

of Niederreiter [15] on the discrepancy

$$\Delta(T) = \sup_{0\le\alpha\le 1} \left|\#\left(\mathcal{F}(T)\cap[0,\alpha]\right) - \alpha F(T)\right|$$

of the Farey fractions.

Since the function

$$f(z) = \begin{cases} z^2 & \text{if } z \in [0, 1/2], \\ 0 & \text{if } z \in (1/2, 1], \end{cases}$$

is of bounded variation, by the classical *Koksma inequality* (see, for example, [16, Theorem 2.9]), we have

$$\begin{aligned} G(T) &= \sum_{\xi\in\mathcal{F}(T)} f(\xi) \\ &= F(T)\int_0^1 f(z)dz + O\left(F(T)\Delta(T)\right) = \frac{1}{24}F(T) + O(T), \end{aligned} \tag{2.8}$$

which together with (2.4) implies the estimate (2.7).

Finally, the asymptotic formulas (2.4) and (2.7) imply Corollary 1.3.

3. Proof of Theorem 1.1

3.1. **Some sums with the Möbius function.** In handling the sums $\mathfrak{M}(T)$ and $\mathfrak{E}(T)$ we often appeal to a result of Gupta [8]:

Lemma 3.1. *For any integer $m \geq 1$, we have*

$$\sum_{\substack{d=1 \\ \gcd(d,m)=1}}^{T} \mu(d) \lfloor T/d \rfloor = \sum_{\substack{d \mid m^\ell \\ d \leq T}} 1,$$

where

$$\ell = \left\lfloor \frac{\log T}{\log 2} \right\rfloor .$$

Note that after changing the order summations, Lemma 3.1 yields

$$\sum_{b \leq T} \sum_{\substack{d \mid b \\ \gcd(d,m)=1}} \mu(d) d = \sum_{\substack{d=1 \\ \gcd(d,m)=1}}^{T} \mu(d) \lfloor T/d \rfloor = \sum_{\substack{d \mid m^\ell \\ d \leq T}} 1.$$

Thus, using it for $m = 1$, we obtain:

Corollary 3.2. *For the following sum we have*

$$\sum_{b \leq T} \sum_{d \mid b} \mu(d) d = 1.$$

We remark, that somewhat related sums have also appeared in the work of Kunik [10, 11]. However, these sums are independent and thus our approach is different and in particular allows for a power saving, while the sums in [11] are estimated with a much weaker saving.

3.2. **Vaaler polynomials.** We define the functions

$$\psi(u) = \{u\} - 1/2 \qquad \text{and} \qquad \mathbf{e}(u) = \exp(2\pi i u).$$

By a result of Vaaler [22] (see also [7, Theorem A.6]), we have:

Lemma 3.3. *For any integer $H \geq 1$ there is a trigonometric polynomial*

$$\psi_H(u) = \sum_{1 \leq |h| \leq H} \frac{a_h}{-2i\pi h} \mathbf{e}(hu)$$

with coefficients $a_h \in [0, 1]$ and such that

$$|\psi(u) - \psi_H(u)| \leq \frac{1}{2H+2} \sum_{|h| \leq H} \left(1 - \frac{|h|}{H+1}\right) \mathbf{e}(hu).$$

We now note that, by Lemma 3.3, we have

$$\sum_{d=1}^{T} \sum_{a/b \in \mathcal{I}(T)} \{da^2/b^2\} M(T/d) = \mathfrak{E}_0 + O(\mathfrak{E}_1 + \mathfrak{E}_2 + T^3/H), \tag{3.1}$$

where

$$\mathfrak{E}_0 = \frac{1}{2} \sum_{a/b \in \mathcal{I}(T)} 1 \times \sum_{d=1}^{T} M(T/d),$$

$$\mathfrak{E}_1 = \sum_{1 \le |h| \le H} |a_h| \left| \sum_{a/b \in \mathcal{I}(T)} \sum_{d=1}^{T} M(T/d) \mathbf{e}(a^2 dh/b^2) \right|,$$

$$\mathfrak{E}_2 = H^{-1} \sum_{1 \le |h| \le H} \left| \sum_{a/b \in \mathcal{I}(T)} \sum_{d=1}^{T} M(T/d) \mathbf{e}(a^2 dh/b^2) \right|$$

(note that T^3/H comes from the contribution of the term with $h = 0$ on the right hand side of the inequality of Lemma 3.3).

Clearly,

$$\mathfrak{E}_0 = -\left(\frac{1}{4} \mathcal{F}(T) + O(1) \right) \sum_{d=1}^{T} M(T/d).$$

Rearranging, for every integer $T \ge 1$, we obtain

$$\sum_{d=1}^{T} M(T/d) = \sum_{k=1}^{T} \mu(k) \lfloor T/k \rfloor = 1, \tag{3.2}$$

by Corollary 3.2. Hence,

$$\mathfrak{E}_0 \ll T^2. \tag{3.3}$$

Substituting (3.3) in (3.1) and combining this with (2.2) we obtain

$$\mathfrak{E}(T) \ll \mathfrak{E}_1 + \mathfrak{E}_2 + T^3/H + T^2, \tag{3.4}$$

3.3. **Bounds of exponential sums.**

Let

$$J = \left\lfloor \frac{\log T}{\log 2} \right\rfloor.$$

We also fix two more positive integer parameters $H \le T$ and $I \le J$, to be determined later.

Define

$$\mathcal{D}_i = \mathbb{Z} \cap \left[2^i, \max\{T, 2^{i+1}\} \right], \qquad i = I, \ldots, J.$$

Using the definition of $\delta(t)$, we have

$$
\begin{aligned}
\mathfrak{E}_1 &\ll \sum_{1\le |h|\le H} \frac{1}{h}\sum_{i=I}^{J} |W_{h,i}| + T^3\delta(T/2^I)\log H,\\
\mathfrak{E}_2 &\ll \frac{1}{H}\sum_{1\le |h|\le H}\sum_{i=I}^{J} |W_{h,i}| + T^3\delta(T/2^I),
\end{aligned} \tag{3.5}
$$

where

$$
W_{h,i} = \sum_{a/b\in\mathcal{I}(T)}\sum_{d\in\mathcal{D}_i} M(T/d)\mathbf{e}(a^2dh/b^2), \qquad i = I,\dots,J.
$$

We fix $i\in[I,J]$ and write

$$
W_{h,i} = \sum_{d\in\mathcal{D}_i} M(T/d)\sum_{b=1}^{T}\sum_{\substack{1\le a\le b/2\\ \gcd(a,b)=1}} \mathbf{e}(a^2dh/b^2).
$$

We estimate $M(T/d)$ trivially as

$$
|M(T/d)| \le T/d \ll T2^{-i},
$$

and obtain

$$
W_{h,i} = T2^{-i}\sum_{d\in\mathcal{D}_i}\sum_{b=1}^{T}\left|\sum_{\substack{1\le a\le b/2\\ \gcd(a,b)=1}} \mathbf{e}(a^2dh/b^2)\right|.
$$

Using that $\#\mathcal{D}_i \ll 2^i$, by the Cauchy inequality, we obtain

$$
|W_{h,i}|^2 \ll T^3 2^{-i}\sum_{d\in\mathcal{D}_i}^{T}\sum_{b=1}^{T}\left|\sum_{\substack{1\le a\le b/2\\ \gcd(a,b)=1}} \mathbf{e}(a^2dh/b^2)\right|^2.
$$

Squaring out and changing the order of summations yields

$$
|W_{h,i}|^2 \ll T^3 2^{-i}\sum_{b=1}^{T}\sum_{\substack{1\le a,c\le b/2\\ \gcd(ac,b)=1}}\sum_{d\in\mathcal{D}_i} \mathbf{e}((a^2-c^2)dh/b^2).
$$

For integer q and u define

$$
\langle u\rangle_q = \|u - q\mathbb{Z}\| = \min_{k\in\mathbb{Z}} |u-kq|
$$

as the distance to the closest integer which is a multiple of q. Then

$$\sum_{d\in\mathcal{D}_i} \mathbf{e}((a^2-c^2)dh/b^2) \ll \min\left\{2^i, \frac{b^2}{\langle (a^2-c^2)h\rangle_{b^2}}\right\}$$

(see [9, Bound (8.6)]). Thus,

$$\begin{aligned}|W_{h,i}|^2 &\ll T^3 2^{-i}\sum_{b=1}^{T}\sum_{1\le a,c\le b}\min\left\{2^i, \frac{b^2}{\langle (a^2-c^2)h\rangle_{b^2}}\right\}\\ &\ll T^3 2^{-i}\sum_{b=1}^{T}\sum_{1\le a,c\le b}\min\left\{2^i, \frac{b^2}{\langle (a^2-c^2)h\rangle_{b^2}}\right\},\end{aligned}$$

where we have dropped the coprimality condition and extended the summation up to b (only for the sake typographical simplicity).

It is convenient to estimate separately the contribution from the diagonal $a=c$, which leads to

$$|W_{h,i}|^2 \ll T^3 2^{-i}\sum_{b=1}^{T}\sum_{1\le a<c\le b}\min\left\{2^i, \frac{b^2}{\langle (a^2-c^2)h\rangle_{b^2}}\right\} + T^5. \tag{3.6}$$

Now for every integer $b\in[1,T]$ we define the set

$$\mathcal{Z}_0(b) = \left\{z\in\mathbb{Z} \ : \ |z|\le 2^{-i}b^2\right\}.$$

Furthermore, for $j=0,\ldots,J$, we define the sets

$$\mathcal{Z}_j(b) = \left\{z\in\mathbb{Z}\cap[-b^2/2, b^2/2] \ : \ 2^{j-i}b^2 < |z| \le 2^{j-i+1}b^2\right\}.$$

Next, we fix some h in the interval $1\le h\le H$ and define the sets:

$$\begin{aligned}\mathcal{A}_j(b) = \{(a,c)\in\mathbb{Z}^2 \ : \ &1\le a<c\le b,\\ &(a^2-c^2)h\equiv z \pmod{b^2} \text{ for some } z\in\mathcal{Z}_j\}.\end{aligned}$$

In particular,

$$\sum_{1\le a<c\le b}\min\left\{2^i, \frac{b^2}{\langle (a^2-c^2)h\rangle_{b^2}}\right\} \ll \sum_{j=0}^{J} 2^{i-j}\#\mathcal{A}_j(b). \tag{3.7}$$

To estimate $\#\mathcal{A}_j(b)$ we note that for each z the congruence

$$(a^2-c^2)h\equiv z \pmod{b^2}$$

puts a^2-c^2 in $\gcd(h,b^2)$ arithmetic progressions modulo b^2. Since $0 < c^2-a^2 < b^2$, each of these progressions, leads to an equation $c^2-a^2=k$ with some positive integer $k\le b^2\le T^2$. Using the classical bound on the divisor function $\tau(m)$ of the integer m (see [9, Equation (1.81)]), we obtain

$$\begin{aligned}\#\mathcal{A}_j(b) &\le \gcd(h,b^2)\#\mathcal{Z}_j(b)\max\{\tau(k) : k \le T^2\}\\ &\le \gcd(h,b^2)\#cZ_j(b)T^{o(1)}\\ &= \gcd(h,b^2)\left(2^{j-i}b^2+1\right)T^{o(1)}\\ &\le \gcd(h,b^2)2^{j-i}T^{2+o(1)},\end{aligned}$$

as $T \to \infty$. Using this in (3.7), we obtain

$$\sum_{1\le a<c\le b} \min\left\{2^i, \frac{b^2}{\langle (a^2-c^2)h\rangle_{b^2}}\right\} \ll J\gcd(h,b^2)T^{2+o(1)} \ll \gcd(h,b^2)T^{2+o(1)},$$

where we ignored the J factor because of the presence of the factor $T^{o(1)}$.

With this notation, we infer from (3.6) that

$$|W_{h,i}|^2 \ll T^{5+o(1)}2^{-i}\sum_{b=1}^{T}\gcd(h,b^2) + T^5. \tag{3.8}$$

3.4. **Concluding the proof.** Since obviously

$$\frac{1}{H}\sum_{1\le |h|\le H}\sum_{i=I}^{J}|W_{h,i}| \le \sum_{1\le |h|\le H}\frac{1}{h}\sum_{i=I}^{J}|W_{h,i}|,$$

we derive from (3.4), (3.5) and (3.8) (and absorbing the term T^2 into T^3/H as $H \le T$), that

$$\mathfrak{E}(T) \ll 2^{-I/2}T^{5/2+o(1)}\Sigma + JT^{5/2}\log H + T^3\delta(T/2^I)\log H + T^3/H, \tag{3.9}$$

where

$$\Sigma = \sum_{1\le |h|\le H}\frac{1}{h}\left(\sum_{b=1}^{T}\gcd(h,b^2)\right)^{1/2}.$$

Writing $h^{-1} = h^{-1/2}h^{-1/2}$ and using the Cauchy inequality, we obtain

$$\Sigma^2 \ll \log H \sum_{1\le |h|\le H}\frac{1}{h}\sum_{b=1}^{T}\gcd(h,b^2).$$

Furthermore, changing the order of summation and collecting together, for each divisor $d \mid b^2$, the values h with $\gcd(h, b^2) = d$, we obtain

$$\begin{aligned}\sum_{1\le |h|\le H} \frac{1}{h}\sum_{b=1}^{T} \gcd(h,b^2) &= \sum_{b=1}^{T}\sum_{1\le |h|\le H}\frac{1}{h}\gcd(h,b^2)\\ &\le \sum_{b=1}^{T}\sum_{d\mid b^2} d \sum_{1\le |k|\le H/d}\frac{1}{dk} = \sum_{b=1}^{T}\sum_{d\mid b^2}\sum_{1\le |k|\le H/d}\frac{1}{k}\\ &\le \log H \sum_{b=1}^{T}\tau(b^2) \ll T(\log H)(\log T)^2.\end{aligned}$$

For the last estimate above, we apply the main result of [12] to the function $f(n) = \tau(n^2)$ which satisfies the conditions of that theorem with $k = 3$. Substituting this in (3.9), we obtain

$$\mathfrak{E}(T) \ll 2^{-I/2}T^{3+o(1)} + T^3\delta(T/2^I)\log H + T^{5/2}(\log H)^2.$$

Choosing now $H = T^{1/2}$ and defining I by the inequalities

$$2^{I-1} < T^{1/2} \le 2^I,$$

we get the conclusion of Theorem 1.1.

4. Proof of Theorem 1.4

4.1. **Preliminaries.** Let $\alpha \ge 0$ and Ψ_α be the (generalized) Dedekind totient function defined by $\Psi_\alpha(1) = 1$ and, for any integer $n \ge 2$

$$\Psi_\alpha(n) = n^\alpha \sum_{d\mid n}\frac{\mu(d)^2}{d^\alpha} = n^\alpha \prod_{p\mid n}\left(1+\frac{1}{p^\alpha}\right).$$

We also define

$$\gamma(\alpha) = \begin{cases} 1, & \text{if } \alpha = 0;\\ 0, & \text{if } \alpha > 0.\end{cases}$$

Note that $\Psi_0 = 2^\omega$ and it is customary to set $\Psi_1 = \Psi$. First we record the following bound

$$\sum_{n\le x}\frac{\Psi_\alpha(n)}{n^\alpha} \ll x\,(\log x)^{\gamma(\alpha)}, \tag{4.1}$$

where the implied constant depends on α, which holds for any real $x \ge 2$. Indeed, from the definition above, we have

$$\sum_{n\le x}\frac{\Psi_\alpha(n)}{n^\alpha} = \sum_{d\le x}\frac{\mu(d)^2}{d^\alpha}\left\lfloor\frac{x}{d}\right\rfloor \ll x\sum_{d\le x}\frac{1}{d^{\alpha+1}} \ll x\,(\log x)^{\gamma(\alpha)}.$$

4.2. **Exponent pairs.** Let $N \geq 1$ be a large integer and let a function $f \in C^\infty[N, 2N]$ have the property that there exists $\mathcal{T} > 0$ such that, for any $x \in [N, 2N]$ and any non-negative integer j, $\left|f^{(j)}(x)\right| \asymp \mathcal{T} N^{-j}$. Let (k, ℓ) be an exponent pair. From [7, Definition p. 31], we have $0 \leq k \leq \frac{1}{2} \leq \ell \leq 1$ and, for any integer $N_1 \in (N, 2N]$

$$\sum_{N < n \leq N_1} \mathbf{e}(f(n)) \ll \mathcal{T}^k N^{\ell - k} + N\mathcal{T}^{-1}.$$

Note that, for any $s \in (0, 1]$, this implies

$$\sum_{N < n \leq N_1} \mathbf{e}(f(n)) \ll \mathcal{T}^k N^{\ell - k} + N\mathcal{T}^{-s}. \tag{4.2}$$

Indeed, if $\mathcal{T} \in (0, 1)$, then $N\mathcal{T}^{-s} > N$ which is the trivial bound, and if $\mathcal{T} \geq 1$, then $N\mathcal{T}^{-s} \geq N\mathcal{T}^{-1}$. The following result is similar to [7, Lemma 4.3].

Lemma 4.1. *Let $d, a \in \mathbb{N}$ and let $T \geq 1$ be sufficiently large. If (k, ℓ) is an exponent pair, then*

$$\begin{aligned}(\log T)^{-1} & \sum_{\substack{2a \leq b \leq T \\ \gcd(b,a)=1}} \psi\left(\frac{a^2 d}{b^2}\right) \\ & \ll \frac{T^2}{d^{1/2}} \frac{\Psi(a)}{a^2} + \begin{cases} \left(d^k a^\ell\right)^{1/(k+1)} \dfrac{\Psi_{\ell-k}(a)}{a^{\ell-k}}, & \text{if } \ell \leq 2k; \\ \left(d^k a^{2k} T^{\ell-2k}\right)^{1/(k+1)} \dfrac{\Psi_{\ell-k}(a)}{a^{\ell-k}}, & \text{if } \ell \geq 2k. \end{cases}\end{aligned}$$

Proof. Let $B \in [2a, T]$. As before, we apply Lemma 3.3, and see that for any integer $H \geq 1$ we have

$$\begin{aligned}\sum_{\substack{B < b \leq 2B \\ \gcd(b,a)=1}} \psi\left(\frac{a^2 d}{b^2}\right) & \ll \frac{B}{H} \frac{\varphi(a)}{a} + \sum_{h \leq H} \frac{1}{h} \left| \sum_{\substack{B < b \leq 2B \\ \gcd(b,a)=1}} \mathbf{e}\left(\frac{a^2 dh}{b^2}\right) \right| \\ & \ll \frac{B}{H} \frac{\varphi(a)}{a} + \sum_{h \leq H} \frac{1}{h} \sum_{\delta \mid a} \mu(\delta)^2 \left| \sum_{B/\delta < b \leq 2B/\delta} \mathbf{e}\left(\frac{a^2 dh}{\delta^2 b^2}\right) \right|.\end{aligned}$$

Now (4.2) with $\mathcal{T} = a^2 dhB^{-2}$ and $s = \frac{1}{2}$ yields

$$\begin{aligned}
&\sum_{\substack{B<b\le 2B\\ \gcd(b,a)=1}} \psi\left(\frac{a^2 d}{b^2}\right)\\
&\quad \ll \frac{B}{H}\frac{\varphi(a)}{a} + \sum_{h\le H}\frac{1}{h}\sum_{\delta\mid a}\mu(\delta)^2\left(B^{\ell-3k}\left(a^2dh\right)^k\delta^{k-\ell} + \frac{B^2}{a\delta(dh)^{1/2}}\right)\\
&\quad \ll \frac{B}{H}\frac{\varphi(a)}{a} + B^{\ell-3k}(Hda^2)^k\frac{\Psi_{\ell-k}(a)}{a^{\ell-k}} + \frac{B^2}{d^{1/2}}\frac{\Psi(a)}{a^2}.
\end{aligned}$$

Assume first that $B^{1-\ell+3k} \ge \left(a^2 d\right)^k$ (note that $1-\ell+3k \ge 0$) and choose

$$H = \left\lfloor \left(B^{1-\ell+3k}a^{-2k}d^{-k}\right)^{1/(k+1)}\right\rfloor.$$

Then $H \ge 1$ and

$$\sum_{\substack{B<b\le 2B\\ \gcd(b,a)=1}} \psi\left(\frac{a^2 d}{b^2}\right) \ll \left(d^k a^{2k}B^{\ell-2k}\right)^{1/(k+1)}\frac{\Psi_{\ell-k}(a)}{a^{\ell-k}} + \frac{B^2}{d^{1/2}}\frac{\Psi(a)}{a^2}.$$

If $B^{1-\ell+3k} > \left(a^2 d\right)^k$, then $\left(d^k a^{2k}B^{\ell-2k}\right)^{1/(k+1)} > B$ and the result is trivially true in that case. We complete the proof by using the usual argument, splitting the whole range of summation into dyadic intervals. □

4.3. **Concluding the proof.** From now on, we assume the Riemann Hypothesis. Theorem 1.4 is a consequence of the following more general result.

Theorem 4.2. *Assume the Riemann Hypothesis and let (k,ℓ) be an exponent pair. For any sufficiently large T*

$$\mathfrak{E}(T) \ll T^{(3k+\ell+2)/(k+1)}\rho(T)(\log T)^2,$$

and if

$$(k,\ell) \ne \left(\frac{1}{2},\frac{1}{2}\right),$$

then the exponent of $\log T$ may be reduced to 1.

Proof. Recalling the definition of $\mathfrak{E}(T)$ in (2.3) and then using (3.2), we derive

$$\begin{aligned}\mathfrak{E}(T) &= -\sum_{a/b\in\mathcal{I}(T)}\sum_{d\le T} M\left(\frac{T}{d}\right)\left\{\frac{da^2}{b^2}\right\}\\ &= -\sum_{a/b\in\mathcal{I}(T)}\sum_{d\le T} M\left(\frac{T}{d}\right)\psi\left(\frac{da^2}{b^2}\right) - \frac{1}{2}\sum_{a/b\in\mathcal{I}(T)}\sum_{d\le T} M\left(\frac{T}{d}\right)\\ &= S(T) + O\left(T^2\right),\end{aligned}$$

where

$$S(T) = -\sum_{a/b\in\mathcal{I}(T)}\sum_{d\le T} M\left(\frac{T}{d}\right)\psi\left(\frac{da^2}{b^2}\right).$$

Let (k,ℓ) be an exponent pair such that $\ell \le 2k$. Lemma 4.1 and the Riemann Hypothesis, implying the inequality (1.4), now yield

$$\begin{aligned}S(T) &\ll T^{1/2}\rho(T)\sum_{d\le T}\frac{1}{d^{1/2}}\sum_{a\le T/2}\left|\sum_{\substack{2a\le b\le T\\ \gcd(b,a)=1}}\psi\left(\frac{a^2d}{b^2}\right)\right|\\ &\ll T^{1/2}\rho(T)\sum_{d\le T}\frac{1}{d^{1/2}}\\ &\qquad\sum_{a\le T/2}\left(\left(d^k a^\ell\right)^{1/(k+1)}\frac{\Psi_{\ell-k}(a)}{a^{\ell-k}} + \frac{T^2}{d^{1/2}}\frac{\Psi(a)}{a^2}\right)\log T\\ &\ll T^{1/2}\rho(T)\log T\\ &\qquad\sum_{d\le T}\frac{1}{d^{1/2}}\left(\left(T^{k+\ell+1}d^k\right)^{1/(k+1)}(\log T)^{\gamma(\ell-k)} + \frac{T^2}{d^{1/2}}\log T\right)\\ &\ll T^{(3k+\ell+2)/(k+1)}\rho(T)(\log T)^{1+\gamma(\ell-k)} + T^{5/2}\rho(T)(\log T)^3\\ &\ll T^{(3k+\ell+2)/(k+1)}\rho(T)(\log T)^{1+\gamma(\ell-k)},\end{aligned}$$

where we have used the bound (4.1). The argument is similar in the case $\ell \ge 2k$. The proof of Theorem 4.2 is complete. □

Now Theorem 1.4 follows from using Theorem 4.2 with the exponent pair

$$(k,\ell) = BA^2\left(\frac{13}{84},\frac{55}{84}\right) = \left(\frac{76}{207},\frac{110}{207}\right),$$

which in turn is derived from the exponent pair (1.6) of Bourgain [2], where A and B denote applications of A- and B-processes, see [7].

Acknowledgements

F. L. was supported in part by grant CPRR160325161141 and an A-rated scientist award both from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency. I. E. S. was supported in part by ARC (Australia) Grant DP140100118.

F.L. and I.S would also like to thank the Max Planck Institute for Mathematics, Bonn, where this work was initiated, for the generous support and hospitality.

References

[1] M. Balazard and A. de Roton, Notes de lecture de l'article 'Partial sums of the Möbius function de Kannan Soundararajan', *Preprint*, 2008, https://arxiv.org/abs/0810.3587. (p. 3)

[2] J. Bourgain, 'Decoupling, exponential sums and the Riemann zeta function', *J. Amer. Math. Soc.*, **30** (2017), 205–224. (pp. 4 and 15)

[3] R. Balasubramanian, S. Kanemitsu and M. Yoshimoto, 'Euler products, Farey series, and the Riemann hypothesis. II', *Publ. Math. Debrecen*, **69** (2006), 1–16. (p. 6)

[4] R. Balasubramanian, F. Luca and D. Ralaivaosaona, 'On the sum of the first n values of the Euler function', *Acta Arith.*, **163** (2014), 199–201. (p. 6)

[5] P. Erdős and H. N. Shapiro, 'On the changes of sign of a certain error function', *Canad. J. Math.*, **3** (1951), 375–384. (p. 5)

[6] L. Fukshansky, P. Guerzhoy and F. Luca, 'On arithmetic lattices in the plane', *Proc. Amer. Math. Soc.*, **145** (2017), 1453–1465. (pp. 1 and 2)

[7] S. W. Graham and G. Kolesnik, *Van der Corput's method of exponential sums*, Cambridge Univ. Press, 1991. (pp. 4, 7, 13, and 15)

[8] H. Gupta, 'A sum involving the Möbius function', *Proc. Amer. Math. Soc.*, **19** (1968), 445–447. (p. 7)

[9] H. Iwaniec and E. Kowalski, *Analytic number theory*, Amer. Math. Soc., Providence, RI, 2004. (pp. 4, 10, and 11)

[10] M. Kunik, 'A scaling property of Farey fractions', *Eur. J. Math.* , **2** (2016), 383–417. (p. 7)

[11] M. Kunik, 'A scaling property of Farey fractions. Part II: convergence at rational points', *Eur. J. Math.*, **2** (2016), 886–896. (p. 7)

[12] F. Luca and L. Tóth, 'The rth moment of the divisor function: an elementary approach', *J. Integer Sequences*, **20** (2017), Art. 17.7.4. (p. 12)

[13] F. Mertens, 'Über einige asymptotische Gesetze der Zahlentheorie,' *Crelle's Journal*, **77** (1874), 289–338. (p. 5)

[14] H. L. Montgomery, 'Fluctuations in the mean of Euler's phi function', *Proc. Indian. Acad. Sci. Math. Sci.*, **97** (1987), 239–245. (p. 5)

[15] H. Niederreiter, 'The distribution of Farey points', *Math. Ann.*, **201** (1973), 341–345. (pp. 4, 5, and 6)

[16] H. Niederreiter, *Random number generation and Quasi–Monte Carlo methods*, SIAM Press, 1992. (p. 6)

[17] A. M. Odlyzko and H. J. J. te Riele, 'Disproof of the Mertens conjecture', *J. Reine Angew. Math.*, **357** (1985), 138–160. (p. 3)

[18] Y.-F. S. Pétermann, 'On an estimate of Walfisz and Saltykov for an error term related to the Euler function', *J. Th. Nombres Bordeaux*, **10** (1998), 203–236. (p. 5)

[19] A. I. Saltykov, 'On Euler's function', *Vestnik Moskov. Univ. Ser. I Mat. Meh.*, **6** (1960), 34–50 (in (Russian). (p. 5)

[20] K. Soundararajan, 'Partial sums of the Möbius function', *J. Reine Angew. Math.*, **631** (2009), 141–152. (p. 3)

[21] D. Suryanarayana and R. Sitaramachandrarao, 'On the average order of the function $E(x) = \sum_{n\le x}\varphi(n) - 3x^2/\pi^2$', *Ark. Mat.*, **10** (1972), 99–106. (p. 6)

[22] J. D. Vaaler, 'Some extremal functions in Fourier analysis', *Bull. Amer. Math. Soc.*, **12** (1985), 183–215. (p. 7)

[23] A. Walfisz, 'Weylsche Exponentialsummen in der neueren Zahlentheorie', Leipzig: B.G. Teubner, 1963. (pp. 3 and 5)

[24] M. Yoshimoto, 'Abelian theorems, Farey series and the Riemann hypothesis', *The Ramanujan J.*, **8** (2004), 131–145. (p. 6)

O.B.: 2 Allée de la combe, 43000 Aiguilhe, France
E-mail address: borde43@wanadoo.fr

School of Mathematics, University of the Witwatersrand, Private Bag X3, Wits 2050, South Africa; Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany; Department of Mathematics, Faculty of Sciences, University of Ostrava, 30 Dubna 22, 701 03 Ostrava 1, Czech Republic
E-mail address: florian.luca@wits.ac.za

Department of Pure Mathematics, University of New South Wales, 2052 NSW, Australia.
E-mail address: igor.shparlinski@unsw.edu.au